M A/M Sc Applied Mathematics, 2nd-Semester, 2016
End-Semester Examination

Course title: Intoduction to Measure Theory
Time allowed: 2 hours
Instructions for the candidates:

Course number: PGAMT2F006T
Maximum Marks: 50

- *The question paper consist of three sections, namely,* Section A, Section B *and* Section C.
- *The* section A *consist of 8 objective type questions, and all the questions are compulsory in this section.*
- *The* section B *consist of 6 short answer type questions, and the candidate has to attempt any 3 questions selecting one question from each unit.*
- *The* section C *consist of 3 long answer type questions, and the candidate has to attempt any 2 questions.*

Section A

1. The set of irrational number is
   (a) a $F_\sigma -$ set (b) not a $F_\sigma -$ set (c) may or may not be a $F_\sigma -$ set (d) none of the these 1.5

2. Let $\mu^\star$ be a metric outer measure on a metric spaces (X, d). Which of the following is false?
   (a) every closed set is measurable (b) every open set is measurable (c) every Borel set is measurable (d) none of the these 1.5

3. Which of the following is false?
   (a) the Cantor ternary set is open (b) the Cantor ternary set is a zero measurable set (c) the Cantor ternary set is uncountable (d) none of the these 1.5

4. Which of the following is true?
   (a) $\chi_{A\cap B} = \chi_A \cdot \chi_B$ (b) $\chi_{A\cup B} = \chi_A + \chi_B$ (c) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$ (d) $\chi_{A\cap B} = \chi_A \cdot \chi_{B^c}$ 1.5

5. Let $\mu^\star$ be the metric outer measure on (X, d) and $A, B \subset X$ such that $\rho(A, B) > 0$. which of the following is true?
   (a) $\mu^\star(A\cup B) < \mu^\star(A) + \mu^\star(B)$ (b) $\mu^\star(A\cup B) = \mu^\star(A) + \mu^\star(B)$ (c) $\mu^\star(A\cup B) > \mu^\star(A) + \mu^\star(B)$ (d) none of the these 1.5

6. Let $\mu^\star$ be a regular outer measure on a set X and let $(A_n : n \in \mathbb{N})$ be an increasing sequence of subsets of X. which of the following is true?
   (a) $\lim_n \mu^\star(A_n) \le \mu^\star(\lim_n A_n)$ (b) $\lim_n \mu^\star(A_n) \ge \mu^\star(\lim_n A_n)$ (c) $\lim_n \mu^\star(A_n) = \mu^\star(\lim_n A_n)$ (d) none of the these 1.5

7. Which of the following is false?
   (a) Lebesgue measure is translation invariant (b) Lebesgue measure is complete (c) Lebesgue measure of an interval is its length (d) none of the these 1.5

8. Which of the following is true?
   (a) [a.e.]⇒[measure] (b) [a.e.]⇒[mean] (c) [a.e.uniform]⇒[measure] (d) [mean]⇒[uniform] 1.5

Section B
Unit I

9. Let X be a non-empty set such that $\#X > 1$. Let

$$\mu^\star(E) = \begin{cases} 0 & \text{if } E = \varphi \\ 1 & \text{if } E \neq \varphi. \end{cases}$$

Then show that $\mu^\star$ is an outer measure, and determine the class of measurable sets. 4

Define $F_\sigma-$Set and $G_\delta-$Set. Prove that if (X, d) is a metric space and d is the topology generated by d, then $\mathfrak{F} \subset \mathfrak{G}_\delta$ and $\mathfrak{G} \subset \mathfrak{F}_\sigma$. 4

Unit II

10. Define measure space and extension of a measure. Prove that Lebesgue measure is complete. Does there exit a subset of $\mathbb{R}$ which is not Lebesgue measurable ? justify your answer. 4

11. Define Borel measurable function. Prove that every measurable function is Lebesgue measurable but converse is true. 4

Unit III

12. Define integration of a positive measurable function. Prove that if s is a positive simple measurable function on $(X, \mathcal{M}, \mu)$ and $\lambda : \mathcal{M} \to [0, \infty]$ defined by $\lambda(E) = \int_E s d\mu$, $E \in \mathcal{M}$, then λ is a measure on $\mathcal{M}$. 4

13. Prove that if f and g are non-negative measurable functions on a measure space $(X, \mathcal{M}, \mu)$, then $\int_A (f+g) d\mu = \int_A f d\mu + \int_A g d\mu$ for $A \in \mathcal{M}$. 4

Section - C

14. Define uncountable set. Prove the following:
    i) if $\mathcal{A}$ is a $\sigma-$algebra of sets in X, then $\mathcal{A}_\sigma = \mathcal{A}_\delta = \mathcal{A}$;
    ii) if $\mathcal{A}$ is an algebra of sets in X, then $\mathcal{A}_\sigma$ need not be a $\sigma-$algebra;
    iii) if $\mathcal{A}$ is an algebra, then for any sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{A}$, we have $\lim_n \sup A_n \in \mathcal{A}$ and $\lim_n \inf A_n \in \mathcal{A}$ for $A \in \mathcal{A}$. 13

15. Let m^* be the Lebesgue outer measure on $\mathbb{R}$. Prove the following:
    i) for every $E \in P(\mathbb{R})$ and $\epsilon > 0$, there exits an open set $E \subset O$ and $m^*(E) \leq m^*(O) \leq m^*(E) + \epsilon$;
    ii) for every $E \in P(\mathbb{R})$, there exits a $G_\delta-$set G in $\mathbb{R}$ such that $G \supset E$ and $m^*(G) = m^*(E)$;
    iii) the Lebesgue outer measure is a Borel regular outer measure. 13

16. If $(X, \mathcal{M}, \mu)$ be a measure space and f, g are measurable functions on X, then prove the following:
    i) if $0 \leq f \leq g$, then $\int_A f d\mu \leq \int_A g d\mu$ for every $A \in \mathcal{M}$;
    ii) if $f \geq 0$, and c is a constant, $0 \leq c < \infty$, then $\int_A (cf) d\mu \leq c \int_A f d\mu$
    iii) if $A \subset B$, and $f \geq 0$, then $\int_A f d\mu \leq \int_B f d\mu$;
    iv) if $f(x) = 0$, for all $x \in A$, then $\int_A f d\mu = 0$, even if $\mu(A) = \infty$;
    v) if $\mu(A) = 0$, then $\int_A f d\mu = \int_X (\chi_A f) d\mu$. 13