M A/M Sc Applied Mathematics, 2nd Semester, 2015-16

End-Semester Examination

Course title: Modern Algebra with applications **Course Code:** PGAMT2C002T

Time allowed: 3 hours **Maximum Marks:** 100

Instructions for the candidates:

- *The question paper consist of three sections, namely,* **Section A**, **Section B** *and* **Section C**.
- *The* **section A** *consist of 10 objetive type questions, and all the questions are compulsory in this section.*
- *The* **section B** *consist of 10 short answer type questions with 2 questions from each unit, and the candidate has to attempt 5 questions selecting one question from each unit.*
- *The* **section C** *consist of 5 long answer type questions with one question from each unit and the candidate has to attempt any 3 questions.*

Section A

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1. The number of groups of order 6 (upto isomorphism) is:

   (a) 1

   (b) 2

   (c) 3

   (d) 4 $1 \cdot 5$

2. Let G be a group such that $|G| = 15$. Then, which of the following statements is false?

   (a) G is abelian.

   (b) G is cyclic.

   (c) $G \simeq \mathbb{Z}/15\mathbb{Z}$.

   (d) None of the above. $1 \cdot 5$

3. The number of groups of order 33 (upto isomorphism) is

   (a) 4

   (b) 2

   (c) 1

(d) 3.

4. For $n \in \mathbb{N}$, let $p(n)$ denote the number of partitions of n. Then $p(4)$ is equal to 1 · 5

(a) 7

(b) 5

(c) 6

(d) 4

5. Let $Aut(\mathbb{Z}/9\mathbb{Z})$ denote the group of automorphisms of $\mathbb{Z}/9\mathbb{Z}$, the group of integers modulo 9. Then the number of elements in $Aut(\mathbb{Z}/9\mathbb{Z})$ is 1 · 5

(a) 4

(b) 6

(c) 2

(d) 5

6. Which of the following statement is false? 1 · 5

(a) $\mathbb{Z}_{12}$, the group of integers modulo 12, has a normal series.

(b) The group $\mathbb{Z}$ of integers has a composition series.

(c) $\mathbb{Z}_6$, the group of integers modulo 6, is solvable.

(d) All of above.

7. Which of the following statement is false? 1 · 5

(a) The polynomial ring $\mathbb{C}[x]$ over the field of complex numbers is a PID.

(b) The polynomial ring $\mathbb{R}[x]$ over the field of real numbers has an identity element.

(c) The ideal $I = \{0\} \subset \mathbb{C}[x]$ is not a prime ideal.

(d) The ideal $I = \langle x^2 + 1 \rangle \subset \mathbb{R}[x]$ is a maximal ideal in $\mathbb{R}[x]$.

8. Which of the following statement is false? 1 · 5

(a) $\mathbb{C}[x]$, the polynomial ring over the field of complex numbers, is a Principal Ideal Domain.

(b) $\mathbb{Z}$ is a Principal Ideal Domain.

(c) $\mathbb{Z}[x]$, the polynomial ring of integers, is a Principal ideal domain.

(d) None of the above.

9. Which of the following statement is false? 1 · 5

(a) $\mathbb{Q}[x]$, the polynomial ring over the field of rational numbers, is a Euclidean domain.

(b) $\mathbb{R}[x]$, the polynomial ring over the field of real numbers, is a PID.

(c) $\mathbb{Z}[x]$ is a UFD, but not a PID.

1 · 5

(d) None of the above.

10. Which of the following statement is true?

(a) $\mathbb{Z}[x]$ is a UFD, but not a PID.

(b) $\mathbb{Z}[x]$ is a PID, but not a Euclidean domain.

(c) $\mathbb{Z}[x]$ is a Eulidean domain.

(d) All of the above.

1 · 5

Section B

Unit - I

1. State first fundamental theorem of group homomorphism. Use it to prove that $\mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$, where $\mathbb{R}$ is group under addition of real numbers, $\mathbb{Z}$ is a group under addition of integers and $\mathbb{S}^1$ is the set of complex numbers of modulus 1 which forms a group under multiplication of complex numbers. 8

2. Let G be a finite group such that $|G| = p^2$, where p is a prime. Prove that G is abelian. Is a group of order 121 is abelian? Justify your answer. 8

Unit - II

3. Prove that $Aut(S_3) \simeq S_3$, where S_3 is the permutation group on 3 symbols $\{1, 2, 3\}$. 8

4. Prove that A_n, $(n \geq 3)$ is generated by all 3-cycles. Moreover, A_n is generated by 3-cycles of the form $(1\ 2\ k)$, $k \geq 3$. 8

Unit - III

5. Prove that a group G is solvabable if and only if $G^{(n)} = \{e\}$ for some n, where $G^{(n)}$ is the nth commutator subgroup of G. 8

6. Define characterstic of a ring. Let R is a commutative ring with identity element 1_R. Then prove that $ch(R) = n$ if and only if $n \cdot 1_R = 0_R$. 8

Unit - IV

7. Prove that if R is a PID, then, for $a_1, a_2, \ldots, a_n \in R$, the greatest common divisor of $a_1, a_2, \ldots, a_n$ exists in R. Moreover, if d is the greatest common divisor of $a_1, a_2, \ldots, a_n$, then $d = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_n$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$. 8

8. Let $f(x) \in \mathbb{F}[x]$ be a quadratic or cubic polynomial, where $\mathbb{F}$ is a field. Then, prove that $f(x)$ is an irreducible polynomial in $\mathbb{F}[x]$ if and only if $f(x)$ has no root in $\mathbb{F}$. 8

Unit -V

9. Let $f(x), g(x) \in R[x]$, where R is a UFD. Then, prove that

$$cont(f(x) \cdot g(x)) = cont(f(x)) \cdot cont(g(x)).$$

8

10. Prove that the polynomial $f(x) = x^p + x^{p-1} + \ldots + x + 1$, where p is a prime number, is irreducible over $\mathbb{Q}$, the field of rational numbers.

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Section - C

1. State and prove Cauchy's Theorem.

3+12

2. State Sylow theorems. Prove that if G is a group of order pq, where $p > q$ are primes, such that q does not divide $p - 1$, then G is cyclic.

6+9

3. Define a prime and maximal ideal with an example in each case. Consider $C[0, 1]$, the ring of continuous real valued functions on closed interval $C[0, 1]$ with pointwise addition and multiplication. For $\frac{1}{2} \in [0, 1]$, define

$$M_{\frac{1}{2}} = \{f \in C[0, 1] : f\left(\frac{1}{2}\right) = 0\}$$

Prove that $M_{\frac{1}{2}}$ is a maximal ideal of $C[0, 1]$.

5+10

4. Define a Euclidean domain. Prove that $\mathbb{Z}[\iota]$ is a Euclidean domain. Compute the units in $\mathbb{Z}[\iota]$.

2+9+4

5. Define an irreducible polynomial and give an example. State and prove Eisenstein criteria for irreducibility of a polynomial. Deduce that for a prime number p, $x^n - p \in \mathbb{Q}[x]$ is irreducible polynomial in $\mathbb{Q}[x]$.

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