M A/M Sc Applied Mathematics
End-Semester Examination

Course title: Complex Dynamical Systems **Course number:** MAMT-409

Time allowed: 3 hours **Maximum Marks:** 100

Instructions for the candidates:

- The question paper consists of three sections, namely, Section A, Section B and Section C.
- The section A consists of 10 objective type questions and all the questions are compulsory in this section.
- The section B consists of 8 short answers type questions, and the candidate has to attempt any 5 questions.
- The section C consist of 10 long answer type questions with 2 questions from each unit, and the candidate has to attempt 5 questions selecting one question from each unit.

Section-A

1. Let $'a'$ be a fixed point of rational map R, then it is called "indifferent fixed point" if

(a) $|R'(a)| = 1$ (b) $|R'(a)| \neq 1$ (c) $|R'(a)| > 1$ (d) $|R'(a)| < 1$

2. If $R(z) = \frac{az+b}{cz+d}$ $where\ ad - bc = 0$ then $R(z)$ is

(a) a map of degree 2 (b) a constant map (c) a zero map (d) none of these

3. Attracting fixed points of a rational map R lie in

(a) Julia set (b) Fatou set (c) closed set (d) none of these

4. If f(z) is not injective in some neighbourhood of z_0 then valency of f(z) at z_0 is

(a) 0 only. (b) 1 only. (c) ∞ only. (d) none of these.

5. If $R(z) = \frac{(1+i)z}{(1-i)z+(1+i)}$, then

(a) $R^n(z) \to 0, as\ n \to \infty$. (b) $R^n(z) \to \infty, as\ n \to \infty$. (c) $R^n(z) \to \infty, as\ n \to 0$. (d) all of these.

6. The fixed point(s) of $f(z) = mz/(nz+1)$

(a) is zero only. (b) is one only. (c) are zero and one. (d) none of these.

7. A point z is called an exceptional point for a map R if

(a) orbit of z is singleton only. (b) orbit of z is an infinite set only.

(c) orbit of z is finite only. (d) z is singular point.

8. A rational map $\frac{z^5+7z^2+7}{z^3+4z}$ has

(a) only two exceptional points (b) infinite number of exceptional points

(c) only one exceptional points (d) no exceptional point

9. If $S = gRg^{-1}$, then for any positive integer $n > 1$

(a) $S^n = g^{-1}R^n g$ (b) $S^n = gRg^{-1}$ (c) $S^n = gR^n g^{-1}$ (d) none of these

10. A mobius map A has ∞ as its fixed point if and only if

(a) A is a polynomial of degree 2. (b) A is a linear map. (c) A is constant. (d) A is a zero map.

Section-B

1. Find a Mobius transformation which maps some specific elements $0,\ 1, \infty$ to $m,\ n,\ p$.
2. If $M(z) = \frac{(1+i)z+(1-i)}{(1-i)z+(1+i)}$, then show that $M^4(z) = z, \forall\, z$.
3. Show that Julia set $J = C_\infty$ or J has empty interior.
4. Define the Fatou set and Julia set of a rational map R with mathematical explanation.
5. If R be a rational map and R^n converges uniformly to some constant on a domain D. Prove that $D \subseteq F(R)$.
6. If z_1 and z_2 are antipodal points of each other on the complex sphere with centre (0, 0, 1/2) and radius 1/2. Show that $z_1.\overline{z_2} = -1$ or $z_2.\overline{z_1} = -1$.
7. Prove that a rational map of degree $m \geq 1$ has precisely $m + 1$ fixed points.
8. Let R be a rational map of degree $d \geq 2$. Then, show that $R(D(J) \subseteq D(J))$, where $D(J)$ be the derived set of Julia set J.

Section-C

Unit -1

1. Find the spherical representation for the extended complex plane and the relation for chordal metric, where the sphere having North Pole (0, 0, 1) and centre (0, 0, 0).
2. Discuss the iterations of the Mobius map $M(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$.

Unit -2

1. Define a rational map and degree of a rational map. Show that a rational map of positive degree d(say) is a d-fold map.

2. Let m be any positive number. Let the Mobius transformation B which satisfies $d(B(0), B(1)) \geq m, d(B(1), B(\infty)) \geq m$ and $d(B(\infty), B(0)) \geq m$. Show that B also satisfies the uniform Lipschitz's condition as $d(B(z), B(w)) \leq \frac{\pi}{m^3} d(z, w)$.

Unit -3

1. If $g: X \to X$ be an onto continuous open map on the topological space X. Let E be any completely invariant set under map g. Then, show that $E', E^0, D(E)$ and $\bar{E}$ are also completely invariant under g.

2. Let f_i be a family of analytic maps in a domain D of the complex sphere and

(i) Each $f \in f_i$ does not take the values $a_{f,} b_{f,} c_{f,}$ in domain D.

(ii) Min $\{d(a_{f,} b_f), \ d(b_{f,} c_f), \ d(c_{f,} a_f)\} \geq m$; where, m is any positive integer. Then, show that f_i is normal in D.

Unit -4

1. Define exceptional points and backward orbit of rational map. Show that the backward orbit of a point is finite if and only if point is an exceptional point.

2. Let R be a rational map with $deg(R) \geq 2$

I. If z is non-exceptional point then $J \subseteq \overline{o(z)}$

II. If $z \in J \Rightarrow J = \overline{o(z)}$

Unit -5

1. If $deg(R) \geq 2$, let F_0 be completely invariant component of F. Then

I. $D(F_0) =$ J

II. F_0 is either simply connected or infinitely connected

III. All other components are simply connected.

IV. F_0 is simply connected iff J is connected.

2. Prove that if Julia set (J) is disconnected then it has infinitely many components.