M A/M Sc Applied Mathematics, 4th-Semester, 2015-16

End-Semester Examination

Course title: Galois Theory **Course number:** MAMT 403

Time allowed: 3 hours **Maximum Marks:** 100

Instructions for the candidates:

- *The question paper consist of three sections, namely,* **Section A**, **Section B** *and* **Section C**.
- *The* **section A** *consist of 10 objetive type questions, and all the questions are compulsory in this section.*
- *The* **section B** *consist of 8 short answer type questions and the candidate has to attempt any 5 questions.*
- *The* **section C** *consist of 10 long answer type questions with 2 questions from each unit, and the candidate has to attempt 5 questions selecting one question from each unit.*

Section A

1. Which of the following statement is false?
   (a) $\mathbb{Q}(\iota) \simeq \mathbb{Q}(-\iota)$.
   (b) $\mathbb{Q}(\pi) \simeq \mathbb{Q}(x)$
   (c) $\mathbb{Q}(\sqrt[5]{3}e^{\frac{2\pi\iota}{5}}) \simeq \mathbb{Q}(\sqrt[5]{3})$
   (d) None of the above. 1

2. Which of the following statement is false?
   (a) $[\mathbb{Q}(e^{\frac{2\pi\iota}{7}}) : \mathbb{Q}] = 6$
   (b) $[\mathbb{Q}(\sqrt[5]{6}) : \mathbb{Q}] = 5$
   (c) $[\mathbb{Q}(\sqrt[7]{2}) : \mathbb{Q}] = 7$
   (d) None of the above. 1

3. Which of the following is a false statement ?
   (a) $\sqrt[5]{2}$ is a constructible number.
   (b) $\sqrt[5]{2}$ is algebraic over $\mathbb{Q}$.
   (c) $\sqrt{2}$ is a constructible number.
   (d) Every constructible real number is algebraic over $\mathbb{Q}$. 1

4. Which of the following statement is false?
   (a) The polynomial $x^2 + x + 1 \in \mathbb{Q}[x]$ has multiple roots.
   (b) The polynomial $x^5 + x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$ is an irreducible polynomial in $\mathbb{Q}[x]$.

(c) The polynomial $x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$ has no multiple root.

(d) The polynomial $x^3 - 6 \in \mathbb{Q}[x]$ is irreducible in $\mathbb{Q}[x]$. 1

5. Which of the following statement is true?

   (a) Every algebraic extension is a finite extension.

   (b) The algebraic closure of finite fields $\mathbb{F}_2$ and $\mathbb{F}_4$, having 2 and 4 elements respectively, is same.

   (c) The algebraic closure of a finite field is finite.

   (d) The finite field $\mathbb{F}_q$ having q elements is algebraically closed. 1

6. Which of the following statement is false?

   (a) $\mathbb{Q}(\sqrt{5})$ is a Galois extension over $\mathbb{Q}$.

   (b) $\mathbb{C}$ is a Galois extension over $\mathbb{R}$.

   (c) $\mathbb{R}$ is a Galois extension over $\mathbb{Q}$.

   (d) $\mathbb{R}$ is a Galois extension over $\mathbb{R}$. 1

7. Consider $f(x) = (x - u_1)(x - u_2)$, the quadratic polynomial, and s_1 & s_2 are elementary symmetric polynomials in u_1 & u_2. Then the discriminant D of $f(x)$ is

   (a) $s_1^2 - 4s_2$.

   (b) $s_1^2 - 2s_2$.

   (c) $s_2^2 - 4s_1$.

   (d) None of the above. 1

8. Which of the following statement is false ?

   (a) The regular pentagon is constructible

   (b) The regular 11-gon is constructible.

   (c) The regular 17-gon is constructible.

   (d) None of the above. 1

9. Let K be a splitting field of cubic polynomial $f(x) \in \mathbb{F}[x]$ such that $[K : \mathbb{F}] = 6$. Then the number of intermediate fields L between $\mathbb{F}$ and K such that L is a Galöis extension over $\mathbb{F}$ is

   (a) 2

   (b) 1

   (c) 4

   (d) 3 1

10. Which of the following is a false statement ?

   (a) $\mathbb{Q}(\sqrt[5]{2}, e^{\frac{2\pi i}{5}})$ is a Galois extenstion over $\mathbb{Q}$.

   (b) $\mathbb{Q}(\sqrt[5]{2}, e^{\frac{2\pi i}{5}})$ is a splitting field of some polynomial with rational coefficients.

   (c) $[\mathbb{Q}(e^{\frac{2\pi i}{5}}) : \mathbb{Q}] = 4$,

   (d) None of the above. 1

Section B

1. Let α is algebraic over a field $\mathbb{F}$. Then, prove that the $[\mathbb{F}(\alpha) : \mathbb{F}]$ is the degree of the irreducible polynomial of α over $\mathbb{F}$.
2. Compute the degree of extension $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ over $\mathbb{Q}$ with all details. 6.
3. Prove that the set of all constructible real numbers form a subfield of $\mathbb{R}$. 6
4. Let K be a finite field. Then prove that the sum of all nonzero elements of K is 0. 6
5. Compute the intermediate subfields of the biquadratic extension $K = \mathbb{Q}(\iota, \sqrt{3})$ over $\mathbb{Q}$. 6
6. Consider the biquadratic extension $\mathbb{Q}(\iota, \sqrt{3})$ over $\mathbb{Q}$. Compute the irreducible polynomial of $\alpha = \iota + \sqrt{3}$ over $\mathbb{Q}$. 6
7. Consider the extension field $K = \mathbb{Q}(\iota, \sqrt[3]{2})$ over $\mathbb{Q}$. Compute the primitive element of K over $\mathbb{Q}$. 6
8. Let p be an odd prime, and let L be a unique quadratic extension of $\mathbb{Q}$ contained in the Cyclotomic field $\mathbb{Q}(e^{\frac{2\pi\iota}{p}})$. Then
$$L = \mathbb{Q}\left(\sqrt{(-1)^{\frac{p-1}{2}}p}\right).$$
6

Section - C

Unit - I

1. (a) Let $\mathbb{F}$ be a field such that $ch(\mathbb{F}) \neq 2$. Then, prove that any extension K of $\mathbb{F}$ of degree 2 can be obtained by adjoining a square root: $K = \mathbb{F}(\delta)$, where $\delta^2 = D$ is an element of $\mathbb{F}$. 6

   (b) Let $\mathbb{F} \subset K \subset L$ be fields. If L is algebraic over K and K is algebraic over $\mathbb{F}$, then L is algebraic over $\mathbb{F}$. 6

2. Let $F \subset L \subset K$ be fields. Then $[K : F] = [K : L][L : F]$. Deduce that if K is a field extension of F of prime degree p and $\alpha \in K \setminus F$, then α has degree p over F and $K = F(\alpha)$. 6+6

Unit - II

3. (a) Let p be a prime. If the regular $p - gon$ can be constructed by ruler and compass, then $p = 2^m + 1$ for some integer m. 6

   (b) Define an algebraic closure of a field. Construct the algebraic closure of a finite field $\mathbb{F}_q$, where $q = p^r$ and p a prime number. 6

4. (a) Prove that $\theta = 20°$ is not constructible by ruler and compass. 6

   (b) Prove that regular $7 - gon$ is not constructible. 6

Unit - III

5. (a) Compute $G(\mathbb{Q}(\sqrt{2}, \iota)/\mathbb{Q})$, the Galois group of biquadratic extension $\mathbb{Q}(\sqrt{2}, \iota)$ over $\mathbb{Q}$. 6

   (b) Let K/F be Galois extension, with Galois group $G = G(K/F)$. Prove that the fixed field of G is F. 6

6. Discuss the Galöis theory of an irreducible cubic polynomial $x^3 + px + q \in \mathbb{F}[x]$ with all details. 12

Unit - IV

7. (a) Prove that every symmetric rational function is a rational function in elementary symmetric functions $s_1, s_2, \ldots, s_n$. 6

   (b) Prove that the discriminant of an irreducible cubic polynomial $f(x) \in \mathbb{F}[x]$ is a square in $\mathbb{F}$ if and only if the degree of its splitting field is 3. 6

8. Let G be a group of automorphisms of a field K of order n and $F = K^G$ be its fixed field. Then $[K : F] = n$. 12

Unit - V

9. Let K/F be a Galois extension, and L be an intermediate field. Let $H = G(K/L)$ be the corresponding subgroup of $G = G(K/F)$.

   (a) Let $\sigma \in G$. Then the subgroup of G which corresponds to the conjugate subfield σL is the conjugate subgroup $\sigma H \sigma^{-1}$,i.e., $G(K/\sigma L) = \sigma H \sigma^{-1}$. 5

   (b) L is a Galois extension of F if and only if H is a normal subgroup of G. 7

10. Let p be a prime number and let $\zeta = e^{\frac{2\pi\iota}{p}}$. Then, prove that

    (a) The Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$ is a cyclic group of order $p-1$.

    (b) For any subfield $\mathbb{F}$ of $\mathbb{C}$, the Galois group of $\mathbb{F}(\zeta)$ over $\mathbb{F}$ is a cyclic group.

12