M A/M Sc Applied Mathematics, 4th-Semester, 2016
End-Semester Examination

Course title: Applied Operator Theory
Time allowed: 3 hours
Course number: MAMT 402
Maximum Marks: 100

Instructions for the candidates:

- *The question paper consist of three sections, namely,* Section A, Section B *and* Section C.
- *The* section A *consist of 10 objective type questions, and all the questions are compulsory in this section.*
- *The* section B *consist of 8 short answer type questions, and the candidate has to attempt any 5 questions.*
- *The* section C *consist of 10 long answer type questions with 5 questions from each unit, and the candidate has to attempt 2 questions selecting one question from each unit.*

Section A

1. If T is symmetric, then T^* — 1
   (a) is symmetric (b) may not be symmetric (c) is bounded (d) none of these

2. If S and T are such that ST is densely defined in H, then — 1
   (a) $(ST)^* \supset T^*S^*$ (b) $(ST)^* \subset T^*S^*$ (c) $(ST)^* = T^*S^*$ (d) none of these

3. If T is symmetry and T_1 is a symmetric extension of T, then — 1
   (a) $T_1 \subset T$ (b) $T \subset T_1$ (c) $T_1 = T$ (d) none of these

4. A linear operator on a finite dimensional complex normed space $X \neq 0$ has — 1
   (a) at least one eigenvalue (b) at most one eigenvalue (c) at least two eigenvalue (d) none of these

5. If $f \in C(X)$, where X is a Hausdorff space, then — 1
   (a) $\sigma(f) \neq$ range f (b) $\sigma(f) =$ range f (c) $\sigma(f) >$ range f (d) none of these

6. The operator $T : l^2 \to l^2$ defined by $Tx = (\frac{\xi_1}{1}, \frac{\xi_2}{2}, \frac{\xi_3}{3}, ...)$ is — 1
   (a) is compact (b) is not compact (c) $\sigma_p(T) \neq \{0\}$ (d) none of these

7. Let $T : D(T) \to l^2$, where $D(T) \subset l^2$ consists of all $x = (\xi_j)$ with only finitely many nonzero terms ξ_j and $y = (\eta_i) = Tx = (j\xi_j)$. This operator T is — 1
   (a) bounded (b) not bounded (c) closed (d) none of these

8. Let A be a complex Banach algebra with identity e. If for an $x \in A$, there are $y, z \in A$ such that $yx = e$ and $xz = e$, then — 1
   (a) x is invertible (b) x is not invertible (c) $x = 0$ (d) none of these

9. A compact linear operator $T : X \to X$ on a normed space X, the set of the eigenvalues of T is — 1
   (a) countable (b) finite (c) empty (d) all of these

10. The righf-shift operator $T : l^2 \to l^2$ defined by $(\xi_1, \xi 2, ...) \to (0, \xi_1, \xi 2, ...)$ — 1
    (a) is unitary (b) is not unitary (c) is not isometric (d) none of these

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Section B

11. Let $X = C[0, 1]$ and define $T : X \to X$ by $Tx = vx$, where $v \in X$ is fixed. Find spectrum $\sigma(T)$. — 6

12. Let X be a complex Banach space, $T \in B(X, X)$ and $\lambda, \mu \in \rho(T)$. Then prove that
    (a) the resolvent R_λ of T satisfies the Hilbert relation or resolvent equation $R_\mu - R_\lambda = (\mu - \lambda)R_\mu R_\lambda$;
    (b) R_λ commutes with any $S \in B(X, X)$ which commutes with T. — 3+3

13. Let X be a complex Banach space, $T : X \to X$ be a linear operator and $\lambda \in \rho(T)$. Assume that (a) T is closed or (b) T is bounded. Then prove that $R_\lambda(T)$ is defined on the whole space X and is bounded. — 3+3

14. Let X be a Banach algebra and $x \in X$. Then $\sigma(x)$ is a non-empty subset of $\mathbb{C}$. — 6

15. Prove that if (x_n) and (y_n) are Cauchy sequences in a normed algebra A, then that $(x_n y_n)$ is Cauchy in A. Also, prove that if $x_n \to x$ and $y_n \to y$, then $x_n y_n \to xy$. 6

16. Let $T : X \to X$ be a compact linear operator on a Banach space X. Then prove that every spectral value $\lambda \neq 0$ of T is an eigenvalue of T. 6

17. Define the sum and product of two linear operators. Prove that if a linear operator T is defined on all of a complex Hilbert space H and satisfies $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $X, y \in H$, then T is bounded. 6

18. State and prove Heisenberg Uncertainty Principle. 6

Section - C
Unit - I

19. State spectral mapping theorem for polynomials. Prove that the eigenvalues of a Skew-Hermitian matrix $A = (\alpha_{ij})$ are purly imaginary or zero and the eigenvectors $x_1, \ldots x_n$ corresponding to different eigenvalues $\lambda_1, \ldots, \lambda_n$ of a linear operator T on a vector space X constitute a linearly independent set. 2+5+5

20. Define a regular value, resolvent set and spectrum. Prove that the resolvent set $\rho(T)$ of a bounded linear operator T on a complex Banach space X is open and the spectrum $\sigma(T)$ is closed. 2+2+8

Unit - II

21. Let $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function and let X be a commutative Banach algebra, $x \in X$ and let $\phi(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $\sigma(\phi(x)) = \phi(\sigma(x)) = \{\phi(\lambda) : \lambda \in \sigma(x)\}$. 12

22. Let A be a complex Banach algebra with identity. Prove that if the set G of all invertible elements of A is an open subset of A, then the subset $M = A - G$ of all non-invertible elements of A is closed. 12

Unit - III

23. Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y. Prove that if (T_n) is uniformly operator convergent i.e., $\|T_n - T\| \to 0$, then the limit operator T is compact. Is this result true if we replace uniform operator convergence by strong operator convergence ? Justify your answer. 12

24. Let X and Y be two normed spaces and let $T : X \to Y$ be a compact linear operator. Suppose that (x_n) in X is weakly convergent. Then prove that (Tx_n) is strongly convergent in Y and has the limit $y = Tx$. 12

Unit - IV

25. Prove that the spectrum $\sigma(T)$ of a self-adjoint linear operator $T : D(T) \to H$ is real and closed; here, H is a complex Hilbert space and $D(T)$ is dense in H. 12

26. Let $T : D(T) \to H$ be a linear operator, where H is a complex Hilbert space and $D(T)$ is dense in H. Then prove that if T is symmetric, its closure $\overline{T}$ exists and is unique. 12

Unit - V

27. Define multiplication operator and differentation operator. Prove that the multiplication operator is self-adjoint and differentation operator is unbounded. 12

28. Prove that the multiplication operator $T : D(T) \to L^2(-\infty, \infty)$ defined by $x \mapsto tx$, where $D(T)$ is a subset of $L^2(-\infty, \infty)$, is self-adjoint. 12
