M A/M Sc Applied Mathematics, Semester - 3rd, 2016-17

End-Semester Examination

Course title: Linear Algebra **Course number:** PGAMT3C002T

Time allowed: 3 hours **Maximum Marks:** 100

Instructions for the candidates:

- *The question paper consist of three sections, namely,* **Section A**, **Section B** *and* **Section C**.
- *The* **section A** *consist of 10 objective type questions, and all the questions are compulsory in this section.*
- *The* **section B** *consist of 10 short answer type questions with 2 questions from each unit, and the candidate has to attempt 5 questions selecting one question from each unit.*
- *The* **section C** *consist of 5 long answer type questions, and the candidate has to attempt any 3 questions.*

Section - A

1. Let P be an $n \times n$ matrix over a field $\mathbb{F}$ such that $PA = AP$; for all $n \times n$ matrices A over $\mathbb{F}$. Then, which of the following statement is true ?

   (a) P is a diagonal matrix.

   (b) $P = I_n$ the $n \times n$ identity matrix.

   (c) $P = \lambda I_n$, where $\lambda \in \mathbb{F}$ and I_n is a $n \times n$ identity matrix.

   (d) None of the above. $1\frac{1}{2}$

2. Which of the following statement is false?

   (a) Every matrix unit is not an invertible matrix.

   (b) For $A, B \in M_n(\mathbb{R})$, $(A - B)(A + B) = A^2 - B^2$.

   (c) Every elementary matrix in an invertible matrix.

   (d) None of the above. $1\frac{1}{2}$

3. If V is a vector space over $\mathbb{C}$ of dimension n, then V as a vector space over $\mathbb{R}$, has dimension

   (a) n

   (b) $2n$

   (c) $n + 1$

   (d) None of the above. $1\frac{1}{2}$

4. Let V be a vector space of dimension n over a field $\mathbb{F}$. Then, which of the following statement is false:

   (a) If a subset $S = \{v_1, v_2, \dots, v_n\}$ of vectors in V is linearly independent, then S spans V.

   (b) If a subset $S = \{v_1, v_2, \dots, v_n\}$ of vectors in V spans V, then S is linearly independent.

   (c) Every subset of V consisting of n vectors is linearly independent.

   (d) None of the above. $1\frac{1}{2}$

5. Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 5 \end{bmatrix}$ be a 2×3 matrix over $\mathbb{R}$. Then, $rank(A)$ is equal to

   (a) 2

   (b) 3

   (c) 1

   (d) None of the above. $1\frac{1}{2}$

6. Which of the following statement is true?

   (a) If $m > n$, then there exist an onto linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

   (b) There exist an onto linear map from $\mathbb{R}^4$ to $\mathbb{R}^4$.

   (c) If $m < n$, then there exist an injective linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

   (d) None of the above. $1\frac{1}{2}$

7. Let A be a $n \times n$ orthogonal matrix and $\{C_1, C_2, \dots, C_n\}$ be the set of columns of A. Then, which of the following is false?

   (a) $\langle C_i, C_j \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker-delta function.

   (b) $\|AX\| = \|X\|$ for all $X \in \mathbb{R}^n$.

   (c) $\det(A) = \pm 1$.

   (d) None of the above. $1\frac{1}{2}$

8. Let A be a real symmetric matrix. Then the eigen values of A are

   (a) positive real numbers.

   (b) purely complex numbers.

   (c) real numbers.

   (d) None of the above. $1\frac{1}{2}$

9. Let A be a $n \times n$ unitary matrix. Then

   (a) eigenvalues of A are real numbers.

   (b) $\det(A) = 1$.

   (c) A is an invertible matrix.

   (d) None of the above. $1\frac{1}{2}$

10. Which of the following statement is false?

(a) The eigenvalues of a skew Hermitian matrix are purely complex numbers of modulus 1.

(b) The eigenvalues of a real symmetric positive definite matrix are positive real numbers.

(c) If A is unitary matrix, then eigenvalues of A are complex numbers of modulus 1.

(d) None of the above. $1\frac{1}{2}$

Section - B

Unit-I

1. Show that elementary matrices are invertible. 8

2. Compute the inverse of matrix $\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$ using row reduction. 8

Unit-II

3. Show that $W = \{A \in M_n(\mathbb{R}) : A \text{ is a symmetric matrix}\} \subset M_n(\mathbb{R})$ is subspace over the field $\mathbb{R}$ of real numbers and compute its dimension. 8

4. Let V be a finite dimensional vector space over a field $\mathbb{F}$, S, T be finite subsets of V such that S is linearly independent and T spans V. Then $|T| \geq |S|$. 8

Unit-III

5. Define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ as follows: For $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$, define

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) := \begin{bmatrix} x_1 + 2x_2 \\ 2x_3 + x_1 \end{bmatrix}.$$

Then show that T is a linear transformation, and compute the matrix of T with respect to ordered basis $B = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $B' = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. 8

6. Use Gram-Schmidt orthonormalisation to construct an orthonormal basis of $\mathbb{R}^3$ from the ordered basis $\left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Unit-IV

7. Define the characterstic polynomial of a linear operator. Use the characterstic polynomial to show that if A is a 2×2 real symmetric matrix, then its eigenvalues are real numbers. 8

8. Let A be a $n \times n$ real matrix. Then show that the following statements are equivalent:

   (a) A is an orthogonal matrix.

   (b) The columns $C_1, C_2, \ldots, C_n$ of A are mutually orthogonal unit vectors in $\mathbb{R}^n$ with respect to standard dot product on $\mathbb{R}^n$.

8

Unit-V

9. Determine the type of conic defined by the polynomial equation $2x_1^2+x_2^2+2x_1+2x_2+1=0$. 8

10. Let $V = \dfrac{\mathbb{C}[t]}{\langle (t-\lambda)^n \rangle}$ be a cyclic $\mathbb{C}[t]$−module. Then the matrix of linear map given by scalar multiplication by t on V as $\mathbb{C}$−vector space with respect to basis $\mathcal{B} = \{w_0, w_1, \ldots, w_{n-1}\}$, where $w_0 = 1 + \langle (1-t)^n \rangle$ and $w_i = (t-\lambda)^i \cdot w_0$; $i = 1, 2, \ldots, n-1$, is a Jordan block J_λ.8

Section - C

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1. (a) Let A and B be $n \times n$ matrices. Then, show that $\det(AB) = \det(A)\det(B)$. 8

   (b) Let A be a $n \times n$ matrix. Then show that $adj(A) \cdot A = \delta \cdot I = A \cdot adj(A)$, where $\delta = \det(A)$. 7

2. (a) Let $S = \{v_1, v_2, \ldots, v_n\}$ be a linearly independent set of vectors in a vector space V over a field $\mathbb{F}$, and $v \in V$. Then $S' = S \cup \{v\} = \{v_1, v_2, \ldots, v_n, v\}$ is a linearly independent subset of V if and only if v does not belong to the subspace spanned by $S = \{v_1, v_2, \ldots, v_n\}$, i.e., $v \notin L(S)$. 9

   (b) Consider

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^3.$$

   Extend S to a basis of $\mathbb{R}^3$. 6

3. (a) Define a linear transformation and give an example of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. 4

   (b) State and prove rank-nullity theorem. 11

4. (a) Define eigenvalue and eigenvector of a linear operator and give an example. 3

   (b) Compute the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ where $\theta \neq 0, \pi$. 12

5. State spectral theorem for a Hermitian operator, and use it to find a 2×2 unitary matrix P such that PAP^* is a real diagonal matrix, where $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$, a 2×2 hermitian matrix. 15