M A/M Sc Applied Mathematics, 3rd-Semester, 2016
End-Semester Examination

Course title: Functional Analysis
Time allowed: 3 hours

Course number: PGAMT3C001T
Maximum Marks: 100

Instructions for the candidates:

- *The question paper consist of three sections, namely,* **Section A**, **Section B** *and* **Section C**.
- *The* **section A** *consist of 10 objective type questions, and all the questions are compulsory in this section.*
- *The* **section B** *consist of 10 short answer type questions with 2 questions from each unit, and the candidate has to attempt 5 questions selecting one question from each unit.*
- *The* **section C** *consist of 5 long answer type questions, and the candidate has to attempt any 3 questions.*

Section A

1. A discrete metric space is
(a) unbounded (b) complete (c) incomplete (d) none of these — 1.5

2. All completions of a normed space are
(a) isometic (b) not equal (c) equal (d) none of these — 1.5

3. Every linear functional on a real vector space is a sublinear ?
(a) True (b) false (c) constant (d) none of these — 1.5

4. Let $X = C[a, b]$. For each $x \in X$, define $\|x\|_2 = (\int_a^b |x(t)|^2 dt)^{\frac{1}{2}}$. Then X is a
(a) normed linear space (b) Banach space (c) complete normed linear space (d) all of these — 1.5

5. The sequence space l_∞ has
(a) Schauder basis (b) no Schauder basis (c) countable basis (d) none of these — 1.5

6. If X_1 and X_2 are subspaces of a vector space X, then the union of X_1 and X_2
(a) is a subspace of X (b) is not a subspace of X (c) may or may not be a subspace of X (d) none of these — 1.5

7. If the product of two linear operators does not exists, then it is
(a) unbounded and linear (b) bounded and nonlinear (c) bounded and linear (d) none of these — 1.5

8. In any infinite dimensional normed linear space, the closed unit ball
(a) is compact (b) is bounded (c) is closed (d) all of these — 1.5

9. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then f is
(a) neither one-to-one nor onto (b) both one-to-one and onto (c) one-to-one but not onto (d) none of the above — 1.5

10. The space l^2 is
(a) a Banach space (b) an inner product space (c) a Hilbert space (d) all of these — 1.5

Section B
Unit I

11. Define Schauder basis and Hamel basis. Prove that $e_n = (\delta_{nj})$ is a Schauder basis for l^p, $1 \leq p < \infty$. — 8

12. Prove that a compact subset M of a metric space is closed and bounded. Is converse true? justify your answer. — 8

Unit - II

13. Prove that if $T : D(T) \to Y$ is a bounded linear operator, where $D(T)$ lies in a normed space X and Y is a Banach space, then T has an extension $T_1 : \overline{D(T)} \to Y$, where T_1 is a bounded linear operator of norm $\|T_1\| = \|T\|$. 8

14. Define Banach space. Prove that if Y is a Banach space, then $B(X, Y)$ is a Banach space. 8

Unit - III

15. Prove that the normed space X of all polynomials with norm defined by $||x|| = \max_i |\alpha_i|$ ($\alpha_0, \alpha_1, \alpha_2, \ldots$ the coefficients of x) is not complete. 8

16. Define adjoint of an operator T. Prove that the adjoint operator $T^\star$ is linear and bounded, and $\|T^\star\| = \|T\|$. 8

Unit - IV

17. Let H_1 and H_2 be Hilbert spaces and $h : H_1 \times H_2 \to \mathbb{K}$ be a bounded sesquilinear form. Then h has a representation $h(x, y) = \langle Sx, y\rangle$, where $S : H_1 \to H_2$ is a bounded linear operator S is uniquely determined by h and has norm $||S|| = ||h||$. 8

18. Let $\{e_1, e_2, ..., e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any vector in H, then prove that

    (i) $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq ||x||^2$.

    (ii) $x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i \perp e_j$ for each j. 8

Unit - V

19. Prove that every bounded linear functional on a Hilbert space H can be represented in terms of the inner product, namely $f(x) = \langle x, z\rangle$ where z depends on f is uniquely determined by f and has norm $||f|| = ||z||$. 8

20. Define self-adjoint and normal operators. If H be a Hilbert space, then prove the following:
    (a) Let A and B be self adjoint. Then $A + B$ is self adjoint.
    (b) Also AB is self adjoint if and only if A and B commutes.
    (c) Let A and B be normal. If A commutes with B^* and B commutes with A^*, then $A + B$ and AB are normal. 8

Section - C

21. (a) Define complete and incomplete normed spaces and prove that the space $(C[0, 1], \|.\|_1)$ is incomplete.

    (b) State F. Riesz's Lemma. Prove that if X and Y are metric spaces and $T : X \to Y$ a continuous mapping, then the image of a compact subset M of X under T is compact. 7+ 8

22. (a) Define dual space. Prove that the dual space of l^p is l^q, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

    (b) Define second algebraic dual space. Prove that $\overline{f}$ is bounded but not linear if f is a bounded linear functional on a complex normed space. 7+ 8

23. (a) Prove that for every fixed x in a normed space X, the functional $g_x(f)$ defined by $g_x(f) = f(x)$ is a bounded linear functional on X so that $g_x \in X''$, and has the norm $\|g_x\| = ||x||$.
    (b) Prove that if (x_n) is a sequence in a normed space X, then strong convergence implies weak convergence with the same limit. Is converse true? 15

24. (a) Prove that if M is a complete subspace Y and $x \in X$ fixed, then $z - y$ is orthogonal to Y.
    (b) Prove that for any subet $M \neq \phi$ of a Hilbert space H, the span of M is dense in H iff $M^\perp = \{0\}$ 7+8

25. (a) Let $T : H \to H$ be a bounded linear operator on a Hilbert space H. Then
    (i) if T is self-adjoint, then $\langle Tx, x\rangle$ is real for all $x \in H$;
    (ii) if H is complex and $\langle Tx, x\rangle$ is real for all $x \in H$, the operator T is self-adjoint.
    (b) Define Legendre, Hermite and Laguerre polynomials with details. 7+8

END (Two pages)