M A/M Sc Applied Mathematics, 4th-Semester, 2016-17

End-Semester Examination

Course title: Galois Theory **Course number:** PGAMT4F005T

Time allowed: 3 hours **Maximum Marks:** 100

Instructions for the candidates:

- *The question paper consist of three sections, namely,* **Section A**, **Section B** *and* **Section C**.
- *The* **section A** *consist of 10 objective type questions, and all the questions are compulsory in this section.*
- *The* **section B** *consist of 10 short answer type questions with 2 questions from each unit, and the candidate has to attempt 5 questions selecting one question from each unit.*
- *The* **section C** *consist of 5 long answer type questions, and the candidate has to attempt any 3 questions.*

Section A

1. Which of the following statement is false?

   (a) $\mathbb{Q}(\sqrt{2}) \simeq \mathbb{Q}(\sqrt{3})$.

   (b) $\mathbb{Q}(\sqrt{\pi}) \simeq \mathbb{Q}(x)$

   (c) $\mathbb{Q}(\sqrt[5]{3}e^{\frac{2\pi\iota}{5}}) \simeq \mathbb{Q}(\sqrt[5]{3})$

   (d) None of the above. 1.5

2. Which of the following statement is false?

   (a) $[\mathbb{Q}(e^{\frac{2\pi\iota}{5}}) : \mathbb{Q}] = 4$

   (b) $[\mathbb{Q}(\sqrt[5]{6}) : \mathbb{Q}] = 5$

   (c) $[\mathbb{Q}(\sqrt[7]{3}) : \mathbb{Q}] = 7$

   (d) None of the above. 1.5

3. Which of the following is a false statement?

   (a) $\sqrt[3]{2}$ is a constructible number.

   (b) $\sqrt[5]{2}$ is algebraic over $\mathbb{Q}$.

   (c) $\sqrt{2}$ is a constructible number.

   (d) None of the above. 1.5

4. Which of the following statements is false?

   (a) The polynomial $x^2 + x + 1 \in \mathbb{Q}[x]$ is an irreducible polynomial over $\mathbb{Q}$.

   (b) The polynomial $x^5 + x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$ is an irreducible polynomial in $\mathbb{Q}[x]$.

   (c) All the roots of the polynomial $x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$ are distinct.

   (d) The polynomial $x^3 - 6 \in \mathbb{Q}[x]$ is irreducible in $\mathbb{Q}[x]$. 1.5

5. Which of the following statement is true?

   (a) Every algebraic extension is a finite extension.

   (b) The algebraic closure of finite fields $\mathbb{F}_4$ and $\mathbb{F}_8$, having 4 and 8 elements respectively, is same.

   (c) The algebraic closure of a finite field is finite.

   (d) Every finite field is algebraically closed. 1.5

6. Which of the following statement is false?

   (a) $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$ is a Galois extension.

   (b) $\mathbb{C}/\mathbb{R}$ is a Galois extension.

   (c) $\mathbb{R}/\mathbb{Q}$ is a Galois extension.

   (d) $\mathbb{R}/\mathbb{R}$ is a Galois extension. 1.5

7. Consider $f(x) = (x - u_1)(x - u_2)$, the quadratic polynomial, and s_1 & s_2 are elementary symmetric polynomials in u_1 & u_2. Then the discriminant D of $f(x)$ is

   (a) $s_1^2 - 4s_2$.

   (b) $s_1^2 - 2s_2$.

   (c) $s_2^2 - 4s_1$.

   (d) None of the above. 1.5

8. Which of the following statement is false ?

   (a) $\cos \frac{2\pi}{5}$ is a constructtible number.

   (b) $\cos \frac{2\pi}{7}$ is a constructible number.

   (c) $\cos \frac{2\pi}{17}$ is a constructible number.

   (d) None of the above. 1.5

9. Let E be a splitting field of cubic polynomial $f(x) \in \mathbb{F}[x]$ such that $[E : \mathbb{F}] = 6$. Then the number of intermediate fields L between $\mathbb{F}$ and K such that L is a Galöis extension over $\mathbb{F}$ is

   (a) 2

   (b) 1

(c) 4

(d) 3 1.5

10. Which of the following is a false statement?

(a) $\mathbb{Q}(\sqrt[5]{2}, e^{\frac{2\pi\iota}{5}})$ is a Galois extenstion over $\mathbb{Q}$.

(b) $\mathbb{Q}(\sqrt[5]{2}, e^{\frac{2\pi\iota}{5}})$ is a splitting field of some polynomial with rational coefficients.

(c) $[\mathbb{Q}(e^{\frac{2\pi\iota}{5}}) : \mathbb{Q}] = 4$,

(d) None of the above. 1.5

Section B

Unit - I

1. Let α is algebraic over a field F. Then show that the $[F(\alpha) : F]$ is the degree of the irreducible polynomial of α over F. 8

2. Compute the degree of extension $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ over $\mathbb{Q}$ with all details. 8

Unit - II

3. Show that a regular pentagon is constructible. 8

4. Let $\mathbb{F}_2$ be a field having two elements. Show that $x^3 + x + 1 \in \mathbb{F}_2[x]$ is an irreducible polynomial and $\frac{\mathbb{F}_2[x]}{\langle x^3 + x + 1 \rangle}$ is a field having eight elements. 8

Unit - III

5. Compute the intermediate subfields of the biquadratic extension $\mathbb{Q}(\iota, \sqrt{3})$ over $\mathbb{Q}$.8

6. Show that if E is the splitting field of an irreducible cubic polynomial $f(x) \in F[x]$, then either $[E : F] = 3$ or $[E : F] = 6$.

Unit - IV

7. Show that every symmetric rational function in $k(u_1, u_2, \ldots, u_n)$ is a rational function in elementary symmetric polynomials $s_1, s_2, \ldots, s_n$. 8

8. Let G be a finite group of automorphisms of a field E and $F = E^G$ be its fixed field. Then $[E : F] = |G|$. 8

Unit - V

9. Let p be a prime number and let $\zeta = e^{\frac{i2\pi}{p}}$. Then the Galois group $Gal(\mathbb{Q}(\zeta)/\mathbb{Q})$ of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$ is a cyclic group of order $p - 1$. 8

10. Let p be an odd prime, and let L be a unique quadratic extension of $\mathbb{Q}$ contained in the Cyclotomic field $\mathbb{Q}(e^{\frac{i2\pi}{p}})$. Then

$$L = \mathbb{Q}\left(\sqrt{(-1)^{\frac{p-1}{2}} p}\right).$$

8

Section - C

1. (a) Let $\mathbb{F}$ be a field such that $ch(\mathbb{F}) \neq 2$. Then, prove that any extension K of $\mathbb{F}$ of degree 2 can be obtained by adjoining a square root: $K = \mathbb{F}(\delta)$, where $\delta^2 = D$ is an element of $\mathbb{F}$. 8

   (b) Let $F \subset L \subset E$ be fields. If E is algebraic over L and L is algebraic over $\mathbb{F}$, then E is algebraic over $\mathbb{F}$. 7

2. (a) Let p be a prime. If the regular $p - gon$ can be constructed by ruler and compass, then $p = 2^{2^m} + 1$ for some integer m. 8

   (b) Define an algebraic closure of a field. Construct the algebraic closure of a finite field $\mathbb{F}_q$, where $q = p^r$ and p a prime number. 7

3. (a) Describe Cardano's method to find a root of a cubic polynomial. 9

   (b) Let E/F be Galois extension, with Galois group $G = Gal(E/F)$. Prove that the fixed field of G is F. 6

4. State and prove primitive element theorem. 15

5. State and prove Fundamental theorem of Galois theory. 15