M A/M Sc Applied Mathematics
End-Semester Examination

Course title: Complex Dynamical Systems   **Course number:** MAMT-409

Time allowed: 3 hours   **Maximum Marks:** 100

Instructions for the candidates:

- The question paper consists of three sections, namely, Section A, Section B and Section C.
- The section A consists of 10 objective type questions and all the questions are compulsory in this section.
- The section B consists of 8 short answers type questions, and the candidate has to attempt any 5 questions.
- The section C consist of 10 long answer type questions with 2 questions from each unit, and the candidate has to attempt 5 questions selecting one question from each unit.

Section-A

1. Let $'a'$ be a fixed point of rational map R, then it is called "indifferent fixed point" if $10 \times 1 = 10$

(a) $|R'(a)| = 1$ (b) $|R'(a)| \neq 1$ (c) $|R'(a)| > 1$ (d) $|R'(a)| < 1$

2. If $R(z) = \frac{az+b}{cz+d}$ where $ad - bc = 0$ then $R(z)$ is

(a) a map of degree 2   (b) a constant map   (c) a zero map   (d) none of these

3. Attracting fixed points of a rational map R lie in

(a) Julia set   (b) Fatou set   (c) closed set   (d) none of these

4. If f(z) is not injective in some neighbourhood of z_0 then valency of f(z) at z_0 is

(a) 0 only.   (b) 1 only.   (c) ∞ only.   (d) none of these.

5. If $R(z) = \frac{(1+i)z}{(1-i)z+(1+i)}$, then

(a) $R^n(z) \to 0, as\ n \to \infty$. (b) $R^n(z) \to \infty, as\ n \to \infty$. (c) $R^n(z) \to \infty, as\ n \to 0$.   (d) all of these.

6. The fixed point(s) of $f(z) = mz/(nz + 1)$

(a) is zero only.   (b) is one only.   (c) are zero and one.   (d) none of these.

7. A point z is called an exceptional point for a map R if

(a) orbit of z is singleton only. (b) orbit of z is an infinite set only.

(c) orbit of z is finite only.   (d) z is singular point.

1

8. A rational map $\frac{z^5+7z^2+7}{z^3+4z}$ has

(a) only two exceptional points   (b) infinite number of exceptional points

(c) only one exceptional points   (d) no exceptional point

9. If $S = gRg^{-1}$, then for any positive integer $n > 1$

(a) $S^n = g^{-1}R^n g$ (b) $S^n = gRg^{-1}$ (c) $S^n = gR^n g^{-1}$ (d) none of these

10. A mobius map A has ∞ as its fixed point if and only if

(a) A is a polynomial of degree 2.   (b) A is a linear map.   (c) A is constant.   (d) A is a zero map.

Section-B

$(5 \times 6 = 30)$

1. Find a Mobius transformation which maps some specific elements 0, 1, ∞ to m, n, p.

2. If $f(z)$ is an injective function in domain $D \Rightarrow v_f(z) = 1 \ \forall \ z \in D$.

   Then, show that the converse of above theorem/statement is not true (give an example).

3. Show that Julia set $J = C_\infty$ or J has empty interior.

4. Define the Fatou set and Julia set of a rational map R with mathematical explanation.

5. If R be a rational map and R^n converges uniformly to some constant on a domain D. Prove that $D \subseteq F(R)$.

6. If z_1 and z_2 are antipodal points of each other on the complex sphere with centre $(0, 0, 1/2)$ and radius $1/2$. Show that $z_1.\bar{z_2} = -1$ or $z_2.\overline{z_1} = -1$.

7. Prove that a rational map of degree $m \geq 1$ has precisely $m + 1$ fixed points.

8. Let R be a rational map of degree $d \geq 2$. Then, show that $R(D(J) \subseteq D(J))$, where $D(J)$ be the derived set of Julia set J.
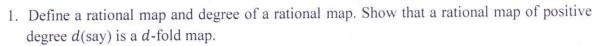
Section-C
Unit -1

$(5 \times 12 = 60)$

1. The Mobius map $A(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$ satisfies the Lipschitz's condition

   $$d_0(A(z), A(w)) \leq \|A\|^2 d_0(z, w) \text{ where, } \|A\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2.$$

2. Discuss the iterations of the Mobius map $M(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$.

Unit -2

1. Define a rational map and degree of a rational map. Show that a rational map of positive degree d(say) is a d-fold map.

2. A non-constant rational map D is conjugate to a polynomial iff $\exists\ z \in C_\infty$ with $R^{-1}(z) = \{z\}$.

Unit -3

1. If N be a rational map and K be a Mobius map. Let $S = KNK^{-1}$. Then, show that $F(S) = K\big(F(N)\big)\ and\ J(S) = K\big(J(N)\big)$.

2. Let f_i be a family of analytic maps in a domain D of the complex sphere and

(i) Each $f \in f_i$ does not take the values $a_f, b_f, c_f,$ in domain D.

(ii) Min $\{d\big(a_f, b_f\big),\ d\big(b_f, c_f\big),\ d\big(c_f, a_f\big)\} \geq m$; where, m is any positive integer. Then, show that f_i is normal in D.

Unit -4

1. Define exceptional points and backward orbit of rational map. Show that the backward orbit of a point is finite if and only if point is an exceptional point.

2. Let R be a rational map with $deg(R) \geq 2$

   I.   If z is non-exceptional point then $J \subseteq \overline{o(z)}$

   II.   If $z \in J \Rightarrow J = \overline{o(z)}$

Unit -5

1. If $deg(R) \geq 2$, let F_0 be completely invariant component of F. Then

   I. $D(F_0) = J$

   II. F_0 is either simply connected or infinitely connected

   III. All other components are simply connected.

   IV. F_0 is simply connected iff J is connected.

2. Prove that if Julia set (J) is disconnected then it has infinitely many components.