M A/M Sc Applied Mathematics, Semester - II, 2017

End-Semester Examination

Course title: Modern Algebra with applications **Course number:** PGAMT2C002T

Time allowed: 3 hours **Maximum Marks:** 100

Instructions for the candidates:

- *The question paper consist of three sections, namely,* **Section A**, **Section B** *and* **Section C**.
- *The* **section A** *consist of 10 objetive type questions, and all the questions are compulsory in this section.*
- *The* **section B** *consist of 10 short answer type questions with 2 questions from each unit, and the candidate has to attempt 5 questions selecting one question from each unit.*
- *The* **section C** *consist of 5 long answer type questions with one question from each unit and the candidate has to attempt any 3 questions.*

Section A

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1. The number of abelian groups of order 6 (upto isomorphism) is:
   (a) 1
   (b) 2
   (c) 3
   (d) 4 $1 \cdot 5$

2. Let G be a group such that $|G| = 15$. Then which of the following statements is false?
   (a) G is abelian.
   (b) G is cyclic.
   (c) $G \simeq \mathbb{Z}/15\mathbb{Z}$.
   (d) None of the above. $1 \cdot 5$

3. The number of groups of order 33 (upto isomorphism) is
   (a) 4
   (b) 2
   (c) 1

(d) 3. 1 · 5

4. For $n \in \mathbb{N}$, let $p(n)$ denote the number of partitions of n. Then $p(5)$ is equal to

(a) 7

(b) 5

(c) 6

(d) 8 1 · 5

5. Let A_n be the set of even permutations of S_n, where S_n denote the group of permutations on $\{1, 2, \ldots, n\}$. Then

(a) A_n is simple, for $n \neq 1$.

(b) A_n is simple, for $n \neq 2$.

(c) A_n is simple, for $n \geq 3$.

(d) A_n is simple, for $n \neq 4$. 1 · 5

6. The series $S_4 \supset A_4 \supset V_4 \supset \{Id\}$ of subgroups of permutation group on $\{1, 2, 3, 4\}$, where A_4 is the set of even permutations of S_4 and $V_4 = \{Id, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$,

(a) is a composition series of S_4.

(b) is a solvable series for S_4.

(c) neither composition nor solvable series for S_4.

(d) None of the above. 1 · 5

7. Consider $C[0, 1]$, the ring of continuous real valued functions on closed interval $[0, 1]$ with pointwise addition and multiplication.Then, which of the following statement is false?

(a) $C[0, 1]$ is a commutative ring.

(b) $C[0, 1]$ has identity element.

(c) $C[0, 1]$ is without zero divisors.

(d) None of the above. 1 · 5

8. Which of the following statement is true?

(a) $\mathbb{Z}[x]$ is not an integral domain.

(b) Every integral domain is a field.

(c) $\mathbb{Z}/p\mathbb{Z}$, where p is prime, is a field.

(d) None of the above. 1 · 5

9. Which of the following statement is false?

(a) Every PID is a Euclidean domain.

(b) Every Euclidean domain is a PID.

(c) Every PID is a UFD.

(d) None of the above. 1 · 5

10. Which of the following statement is true?

(a) Every UFD is a PID.

(b) $\mathbb{Z}[x]$ is a PID.

(c) $\mathbb{Z}[x, y]$ is a UFD, but not a PID.

(d) All of the above. 1 · 5

Section B

Unit - I

1. Define $\varphi : (\mathbb{R}, +) \longrightarrow (\mathbb{S}^1, \cdot)$ given by $\varphi(t) = e^{i2\pi t}; t \in \mathbb{R}$,where $\mathbb{R}$ is group under addition of real numbers, $\mathbb{Z}$ is a group under addition of integers and $\mathbb{S}^1$ is the set of complex numbers of modulus 1 which forms a group under multiplication of complex numbers. Show that φ is onto homomorphism with $Ker(\varphi) = \mathbb{Z}$. Deduce that $\mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$. 8

2. Let G be a finite group such that $|G| = p^2$, where p is a prime. Prove that G is abelian. Is a group of order 169 is abelian? Justify your answer. 8

Unit - II

3. Prove that $Aut(S_3) \simeq S_3$, where S_3 is the permutation group on 3 symbols $\{1, 2, 3\}$. 8

4. Let $\tau = (i_1\ i_2\ \dots\ i_r) \in \mathfrak{S}_n$ be an r-cycle and $\sigma \in \mathfrak{S}_n$ be any permutation. Then $\sigma\tau\sigma^{-1} = (\sigma(i_1)\ \sigma(i_2)\ \dots\ \sigma(i_r))$ is also an r-cycle. 8

Unit - III

5. Prove that a group G is solvabable if and only if $G^{(n)} = \{e\}$ for some n, where $G^{(n)}$ is the nth commutator subgroup of G. 8

6. Let R be a commutative ring with identity 1_R. Prove that an ideal $M \subset R$ is a maximal ideal in R if and only if R/M is a field. 8

Unit - IV

7. Prove that every ideal in a Euclidean domain is a principal ideal. 8

8. Show that 2 is not a prime element in the ring $\mathbb{Z}[\sqrt{d}]$, where $d \in \{-2, -1, 2, 3\}$. 8

Unit - V

9. Let R be a UFD and $f(x), g(x) \in R[x]$ are primitive polynomials. Prove that the product $f(x)g(x)$ is also a primitive polynomial. 8

10. Prove that the cyclotomic polynomial $f(x) = x^{p-1} + x^{p-2} + \ldots + x + 1$, where p is a prime number, is an irreducible polynomial over the field $\mathbb{Q}$ of rational numbers. 8

Section - C

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1. State and prove Cauchy's Theorem. 3+12

2. State Sylow theorems. Prove that if G is a group of order pq, where $p > q$ are primes, such that q does not divide $p - 1$, then G is cyclic. 6+9

3. Consider the map $\varphi : C[0,1] \longrightarrow \mathbb{R}$ given by $\varphi(f) = f\left(\frac{1}{2}\right); f \in C[0,1]$, where $C[0,1]$ is the ring of continuous real valued functions on the closed interval $[0,1]$. Show that φ is an onto ring homomorphism with $Ker(\varphi) = \{f \in C[0,1] : f\left(\frac{1}{2}\right) = 0\}$. Deduce that $Ker(\varphi)$ is a maximal ideal of $C[0,1]$. 15

4. (a) Define a Euclidean domain. Show that the polynomial ring $F[x]$ over a field is a Euclidean domain. 2+8

   (b) Compute the units in the ring $\mathbb{Z}[i]$ of Gaussian integers. 5

5. State and prove Gauss Theorem. Deduce that the polynomial ring $\mathbb{Z}[x, y]$ over $\mathbb{Z}$ in two variables x, y is a UFD. 12+3